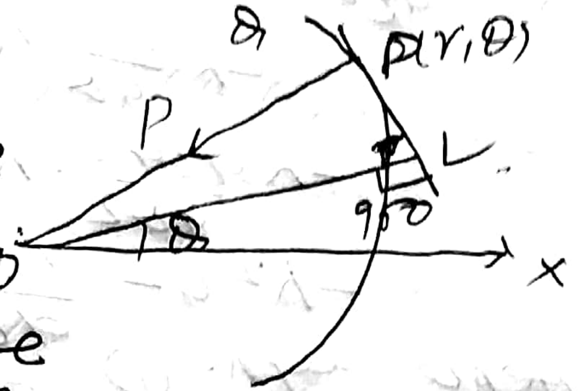
B.Sc. (Maths) part II

paper IV

Topic :- Motion of particle under Central force (dynamics)

Theorem :- A particle moves in a plane with acceleration which is always directed to a fixed point O in the plane, obtain the differential equation of its path.

proof :-



Let us take O as origin and the fixed line OX as initial line. Let the ~~~~ position of particle at time t be P(r, θ) & P be the acceleration of the particle directed towards O, we have

$$\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P \quad (1)$$

Since there is no acceleration perpendicular to P Therefore

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$$

∴ on integration

$$\frac{d\theta}{dt} = \text{const.} = h \text{ (say)}$$

i.e. $\frac{d\theta}{dt} = \frac{h}{r^2} = hu^2$ where $r = \frac{1}{u}$

~~Now $\frac{dr}{dt} = \frac{h}{\theta} = h$~~

Now $\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt}$

$$= -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta}$$

and $\frac{d^2r}{dt^2} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right)$

$$= -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$= -h\frac{d^2u}{d\theta^2}\cdot hu^2$$

$$= -h^2u^2\frac{d^2u}{d\theta^2}$$

Hence eqn. 1 becomes

$$-h^2\theta^2\frac{d^2u}{d\theta^2} - \frac{1}{u}h^2u^2 = -P$$

$$\therefore P = h^2u^2\left(u + \frac{d^2u}{dt^2}\right) \quad \text{—} \ 2$$

From O draw OL (= p) perpendicular to the tangent at P.

Then $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$

(from diff. Cal.)

differentiating both sides with respect to θ we get

$$-\frac{2}{p^3}\cdot\frac{dp}{d\theta} = 2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2}$$

or $$-\frac{1}{p^3}\frac{dp}{dr}\frac{dr}{d\theta} = \left(u+\frac{d^2u}{d\theta^2}\right)\frac{du}{d\theta}$$

or $$-\frac{1}{p^3}\frac{dp}{dr}\cdot\left(-\frac{1}{u^2}\frac{du}{d\theta}\right) = \left(u+\frac{d^2u}{d\theta^2}\right)\frac{du}{d\theta}$$

as $r=\frac{1}{u}$ $\therefore \frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}$

i.e. $$\frac{1}{p^3}\frac{dp}{dr} = u^2\left(u+\frac{d^2u}{d\theta^2}\right) = \frac{P}{h^2}$$

Hence $P = \frac{h^2}{p^3}\frac{dp}{dr}$ from (2)

This gives (p,r) equation of the path.